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THE GOVERNING EQUATIONS OF A THIN ELASTIC STRESSED BEAM WITH A PERIODIC STRUCTURE†

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The equilibrium equations, which govern the equations and boundary conditions for a thin elastic stressed beam with a periodic structure, are derived by the method of averaging. Unlike previous publications [1–3], initial stresses comparable with Young's modulus of the beam material are considered. © 1999 Elsevier Science Ltd. All rights reserved.

Previously [1, 2] the three-dimensional problem of the theory of elasticity was converted into a one-dimensional problem of the theory of beams when there are no initial stresses. The limiting one-dimensional problem was obtained by the method of asymptotic expansions in the small-diameter domain.

Note that the shearing force, which plays an important role in classical beam theory, was eliminated in [1] from the equilibrium equations without obtaining any expression for it in terms of the axial deformation and curvature (as in classical theory). In this connection, only the "axial force, moments-axial deformation, curvature, and torsion" relation must be referred to the governing equations of stress-free beams. A different situation arises when there are initial stresses. It was pointed out in [4, 5] that the shearing forces in a stressed beam are asymmetrical, so that they cannot be eliminated by the classical method. An analysis of the shearing forces led to the derivation of the classical stability equations by methods of asymptotic analysis [4], and to stability equations when moments of the initial stresses are present [5]. The following order of magnitudes were assumed in [4, 5]: elasticity constants of the order of ε^{-4} (which guarantees non-zero bending stiffness as $\varepsilon \to 0$), initial stresses of the order of ε^{-2} [4] (which corresponds to a non-zero axial force), and of the order of ε^{-3} (which corresponds to non-zero moments). It was pointed out in [4, 5] that the initial stresses have no effect on the stiffness of the beam during torsion, which contradicts the results obtained for a thick rod in [6].

In this paper we consider the problem for initial stresses of the order of ε^{-4}. We will show, in particular, that it is in precisely this case that the relation described earlier in [6] arises.

1. FORMULATION OF THE PROBLEM

Consider a region G_ε, obtained by periodic repetition of a certain periodicity cell P_ε along the Ox_1 axis from $-a$ to a (see Fig. 1). The characteristic size of the periodicity cell (which is identical with the beam thickness) $\varepsilon \ll 1$, which is formalized in the form $\varepsilon \to 0$. As $\varepsilon \to 0$ the region G_ε, contracts to the section $[-a, a]$—the beam.

The equilibrium problem for a body with initial stresses has the form [6]

$$\int\limits_{G_\varepsilon} \sigma_{ij} \partial v_i / \partial x_j d\mathbf{x} = 0 \tag{1.1}$$

for any function $\mathbf{v} \in V(G_\varepsilon) = \{\mathbf{v} \in H^1(G_\varepsilon) : \mathbf{v}(\mathbf{x}) = 0 \text{ when } x_1 = -a, a\}$ (for more detail on these classes of functions see [7]). Here, the relation between the actual stresses σ_{ij}^ε, displacements $\mathbf{u}^\varepsilon$ and the initial stresses σ_{jl}^* has the form

$$\sigma_{ij}^\varepsilon = \varepsilon^{-4}(a_{ijkl}(\mathbf{y}) + b_{ijkl}(x_1, \mathbf{y}))\partial u_k^\varepsilon / \partial x_1 = \varepsilon^{-4} A_{ijkl}(\mathbf{y})(x_1, \mathbf{y})\partial u_k^\varepsilon / \partial x_l \tag{1.2}$$

where $\varepsilon^{-4} a_{ijkl}(\mathbf{y})$ are the components of the elasticity constants tensor, $b_{ijkl}(x_1, \mathbf{y}) = \sigma^*_{jl}(x_1, \mathbf{x}/\varepsilon)\delta_{ik}$; $\sigma_{il}^{*(p)}(x_1, \mathbf{x}/\varepsilon)$ are the components of the tensor of the initial stresses, δ_{ik} is the Kronecker delta and $\mathbf{y} = \mathbf{x}/\varepsilon$ are fast (local) variables.

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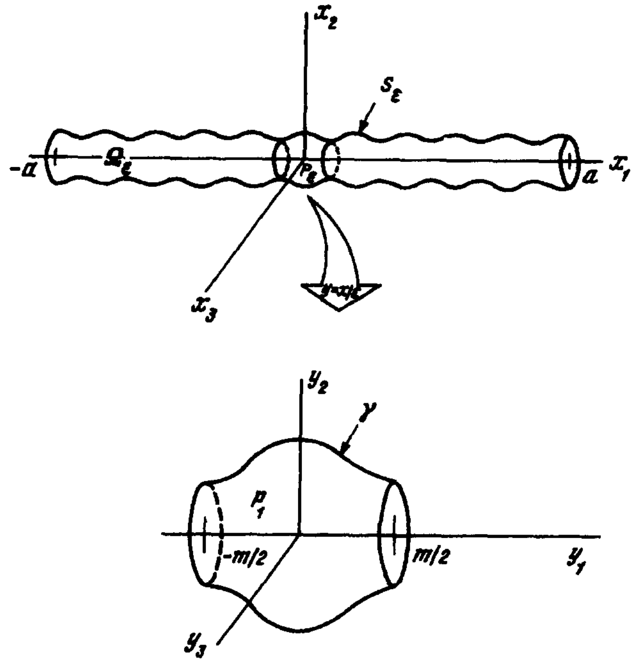

Fig. 1.

It was shown in [4], that initial stresses of lower order lead to a stability loss if the axial force or the moments are non-zero. We therefore impose the following conditions on the stresses

$$\left\langle \sigma_{i1}^{*} \right\rangle = 0, \quad \left\langle \sigma_{i1}^{*} y_{\alpha} \right\rangle = 0, \quad i = 1,2,3; \quad \alpha = 2,3 \quad \left(\langle . \rangle = 1/m \int_{Y} \cdot \, d\mathbf{y}\right) \tag{1.3}$$

where m is the length of the periodicity cell Y along the Oy_1 axis (see Fig. 1) and $Y = \{\mathbf{x}/\varepsilon : \mathbf{x} \in P_\varepsilon\}$ is the periodicity cell in dimensionless variables.

If conditions (1.3) are not satisfied we must use the models proposed previously in [4, 5].

2. ASYMPTOTIC EXPANSION

We will take the asymptotic expansion for the variables of the stresses $\mathbf{u}^\varepsilon$ and the trial function $\mathbf{v}$ in the form [1]

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(x_1) + \varepsilon \mathbf{u}^{(1)}(x_1, \mathbf{y}) + \ldots = \mathbf{u}^{(0)}(x_1) + \sum_{n=1}^{\infty} \varepsilon^n \mathbf{u}^{(n)}(x_1, \mathbf{y}) \tag{2.1}$$

$$\sigma_{ij}^\varepsilon = \sum_{m=-4}^{\infty} \varepsilon^m \sigma_{ij}^{(m)}(x_1, \mathbf{y}) \tag{2.2}$$

$$\mathbf{v} = \mathbf{v}^{(0)}(x_1) + \varepsilon \mathbf{v}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{v}^{(n)}(x_1, \mathbf{y}) \tag{2.3}$$

Here $\mathbf{y} = \mathbf{x}/\varepsilon$ are the fast variables and x_1 is the slow variable. The functions on the right-hand side of (2.1)–(2.3) are assumed to be periodic in y_1 with period m.

Remark 1. The derivative of a function of the form $Z(x_1, \mathbf{y})$ is calculated by replacing the differentiation operator in accordance with the rule [1]

$$\partial Z / \partial x_1 = Z_{,1x} + \varepsilon^{-1} Z_{,1y}, \quad \partial Z / \partial x_\alpha = \varepsilon^{-1} Z_{,\alpha y} \quad (\alpha = 2,3)$$

Here and henceforth the Latin subscripts takes the values 1, 2, 3, the Greek subscripts take the values –2, 3, m = –4, –3, . . . , and n = 0, 1, ...

Substituting (2.1) and (2.2) into (1.2) and taking Remark 1 into account we obtain

$$\sum_{m=-4}^{\infty} \varepsilon^m \sigma_{ij}^{(m)} = \sum_{n=0}^{\infty} \varepsilon^{n-4} A_{ijkl}(x_1, \mathbf{y})(u_{k,1x}^{(n)} \delta_{l1} + \varepsilon^{-1} u_{k,ly}^{(n)}) \tag{2.4}$$

Here and henceforth we will use the notation: $,1y = \partial/\partial y_1$ and $,\alpha x = \partial/\partial x_\alpha$.

Equating terms of like powers of ε in (2.4) we obtain

$$\sigma_{ij}^{(m)} = a_{ijkl}(x_1, \mathbf{y}) u_{k,1x}^{(m+4)} + b_{ijkl}(x_1, \mathbf{y}) u_{k,1x}^{(m+4)} + a_{ijkl}(x_1, \mathbf{y}) u_{k,ly}^{(m+5)} + b_{ijkl}(x_1, \mathbf{y}) u_{k,ly}^{(m+5)} \tag{2.5}$$

Note that $\sigma_{ij}^{\varepsilon}$ defined in (1.2) and σ_{ij}^{m} defined in (2.5) are asymmetric with respect to ij.

Substituting (2.2) and (2.3) into the equilibrium equation (1.1) and taking Remark 1 into account we obtain

$$\sum_{m=-4}^{\infty} \sum_{n=0}^{\infty} \int_{G_1} \varepsilon^{m+n} (\sigma_{i1}^{(m)} v_{i,1x}^{(n)} + \varepsilon^{-1} \sigma_{ij}^{(m)} v_{i,jy}^{(n)}) dz = 0 \tag{2.6}$$

$$G_1 = \{\mathbf{z} = (x_1, y_2, y_3) = (x_1, x_2/\varepsilon, x_3/\varepsilon) : \mathbf{x} \in P_\varepsilon\}$$

3. THE EQUILIBRIUM EQUATIONS

As was pointed out in [8, 9], the equilibrium equations when using the asymptotic method are obtained independently of the "stress–strain" relation, i.e. in the case considered, of (1.2). The equations for the axial forces $N_{ij}^{(m)} = \langle \sigma_{ij}^{(m)} \rangle$ and the moment $\langle M_{j\alpha}^{(m)} = \sigma_{j1}^{(m)} y_\alpha \rangle$ were obtained for beams in [1]. We will give the equilibrium equations taken from [1], which we will use below. These are

$$N_{11,1x}^{(-4)} = 0 \tag{3.1}$$

$$-M_{i\alpha,1x}^{(-4)} + N_{i\alpha}^{(-3)} = 0, \quad N_{\alpha 1,1x}^{(-3)} = 0 \tag{3.2}$$

$$-M_{i\alpha,1x}^{(-3)} - N_{i\alpha}^{(-2)} = 0, \quad N_{\alpha 1,1x}^{(-2)} = 0 \tag{3.3}$$

4. THE GOVERNING EQUATIONS FOR A BEAM (THE RELATION BETWEEN THE FORCES AND MOMENTS AND THE DEFORMATION CHARACTERISTICS)

To obtain the equations considered below we will use relation (2.6) with an appropriate choice of m and n and trial function $\mathbf{v}$. Note that the equations for $\sigma_{ij}^{(-4)}$ and $\sigma_{ij}^{(-3)}$ are also independent of the "stress–strain" relation (see [1] for more detail), while the specific feature of the problem begins to manifest itself with the expressions for $\sigma_{ij}^{(-4)}$ and $\sigma_{ij}^{(-3)}$ obtained from (2.5). We will put $m = -4$ and $n = 1$ in (2.6) and take the trial function in the form $\mathbf{v} = \varepsilon \mathbf{v}(\mathbf{y})$. This leads to the following problem (for more detail see [1])

$$\sigma_{ij,iy}^{(-4)} = 0 \text{ in } G_1, \quad \sigma_{ij}^{(-4)} n_j = 0 \text{ on } \gamma_1 \tag{4.1}$$

where $\mathbf{n}$ is the unit vector of the normal to the side surface γ_1 of region G_1.

From (2.5) and the definition of the quantities A_{ijkl} and b_{ijkl} we can write

$$\sigma_{ij}^{(-4)} = A_{ijkl}(x_1, \mathbf{y}) u_{k,ly}^{(1)} + a_{ijk1}(x_1, \mathbf{y}) u_{k,1x}^{(0)} + \sigma_{j1}^{*}(x_1, \mathbf{y}) \delta_{ik} u_{k,1x}^{(0)} \tag{4.2}$$

Proposition 1. Suppose the initial stresses satisfy the equations

$$\partial \sigma_{ij}^{*} / \partial x_j = 0 \text{ in } G_\varepsilon, \quad \sigma_{ij}^{*} n_j = 0 \text{ on } S_\varepsilon \tag{4.3}$$

Then

$$\sigma_{ij,jy}^{*} = 0 \text{ in } Y, \quad \sigma_{ij}^{*} n_j = 0 \text{ on } \gamma \tag{4.4}$$

where γ is the side surface of the periodicity cell Y.

Equation (4.4) is obtained by substituting into (4.4) expansions of the form (2.2) for the initial stresses.

Proposition 2. If the initial stresses satisfy Eqs (4.3), then $\langle \sigma^*_{i\alpha} \rangle = 0$.

According to Proposition 1 the initial stresses satisfy Eqs (4.4). Multiplying the equation from (4.4) by y_α and integrating by parts over the periodicity cell Y, we obtain the required proposition.

Proposition 3. For any function $Z(y)$ that is periodic in y_1 with period m, we have the equality $\langle \sigma^*_{ij} Z_{,jy} \rangle = 0$. To verify this it is sufficient to multiply the equation from (4.4) by $Z(y)$ and integrate by parts over the periodicity cell Y taking into account the boundary condition from (4.4) and the periodicity conditions.

Remark 2. The functions of the variable x_1 play the role of parameters in the problem in the variables y [10]. The same occurs when integrating with respect to the variable y.

Problem (4.1) is written in the variables y. Taking Remark 2 into account, the solution of problem (4.1) can be represented in the form

$$\mathbf{u}^{(1)} = \mathbf{N}_\sigma^{p1}(\mathbf{y}) u_{p,1x}^{(0)}(x_1) + y_B s_\beta \mathbf{e}_\beta \phi(x_1) + \mathbf{V}(x_1) \tag{4.5}$$

where $\phi(x_1)$ is an as yet undefined function (which has the meaning of the torsion of the beam), $\mathbf{V}(x_1)$ is an as yet undefined function (having the meaning of the axial displacement), $s_1 = 0$, $s_2 = -1$ and $s_3 = 1$, the subscript $B = 2$ if $\beta = 3$ and $B = 3$ if $\beta = 2$, $\{e_i\}$ are basis unit vectors of a standard rectangular system of coordinates, and $\mathbf{N}_\sigma^{p1}(\mathbf{y})$ is the solution of the cell problem

$$(A_{ijkl}(\mathbf{y}) N_{\sigma k,ly}^{p1} + a_{ijp1}(\mathbf{y}) + b_{ijp1}(\mathbf{y}))_{,jy} = 0 \ \text{ in } Y \tag{4.6}$$

$$(A_{ijkl}(\mathbf{y}) N_{\sigma k,ly}^{p1} + a_{ijp1}(\mathbf{y}) + b_{ijp1}(\mathbf{y})) n_j = 0 \ \text{ on } \gamma$$

$\mathbf{N}_\sigma^{p1}(\mathbf{y})$ is periodic in y_1 with period m.

Proposition 4. Cell problem (4.6) has the particular solution $\mathbf{N}_\sigma^{\alpha 1}(\mathbf{y}) = -y_\alpha \mathbf{e}_1$, and the homogeneous problem corresponding to (4.6) has the solution $\mathbf{X}(\mathbf{y}) = y_B s_\beta \mathbf{e}_\beta$.

Proposition 4 is important in view of the fact that it is precisely the presence of explicit solutions which enables formula (4.5) to be obtained.

We first note that $b_{ijp1}(\mathbf{y})_{jy} = \sigma^* b_{1j,jy} \delta_{ip} = 0$ in Y by virtue of (4.4), and $b_{ijp1}(\mathbf{y})_{nj} = \sigma^*{}_{1j} n_j \delta_{ip} = 0$ on γ also by virtue of (4.4).

Further we will verify that $-y_\alpha \mathbf{e}_1$ is a solution of cell problem (4.6). Substituting $-y_\alpha \mathbf{e}_1$ into the left-hand side of the equation from (4.6), taking into account the definition of the quantities A_{ijkl} and b_{ijkl} we obtain $(-a_{ij1\alpha}(\mathbf{y}) + b_{ij1\alpha}(\mathbf{y})] + a_{ij\alpha 1}(\mathbf{y}))_{,jy} = (-a_{ij1\alpha}(\mathbf{y}) + a_{ij\alpha 1}(\mathbf{y}))_{,jy} - b_{ij1\alpha,jy}$. The latter expression is equal to zero. In fact, $-a_{ij1\alpha}(\mathbf{y}) + a_{ij\alpha 1}(\mathbf{y}) = 0$ by virtue of the symmetry of the elasticity constants [3], while $b_{ij1\alpha jy} = 0$ by virtue of (4.4). It can similarly be verified that, by virtue of (4.4), $(-[a_{ij1\alpha}(\mathbf{y}) + b_{ij1\alpha}(\mathbf{y})] + a_{ij\alpha 1}(\mathbf{y}))n_j = 0$ on γ.

We will verify that the function $y_B s_\beta \mathbf{e}_\beta$ is a solution of homogeneous cell problem (4.6). Substituting $y_B s_\beta \mathbf{e}_\beta$ into the homogeneous equation corresponding to the equation from (4.6) and taking the definition of the quantities A_{ijkl} and b_{ijkl} into account, we obtain

$$(a_{ij\beta B} + b_{ij\beta B} s_\beta)_{,jy} = (a_{ij23} - a_{ij32})_{,jy} + b_{ij\beta B,jy} s_\beta = 0$$

in view of the symmetry of the elasticity constants [3] and (4.4). Similarly, for the function $y_B s_\beta \mathbf{e}_\beta$ and the homogeneous boundary condition from (4.6), taking into account the symmetry of the elasticity constants [3] and Eqs (4.4), we obtain that

$$(a_{ij\beta B} + b_{ij\beta B} s_\beta) n_j = 0 \ \text{ on } \gamma$$

As a result, (4.5) can be written in the form

$$\mathbf{u}^{(1)} = \mathbf{N}_\sigma^{11}(\mathbf{y}) u_{1,1x}^{(0)}(x_1) - y_\alpha \mathbf{e}_1 u_{\alpha,1x}^{(0)}(x_1) + y_B s_\beta \mathbf{e}_\beta \phi(x_1) + \mathbf{V}(x_1) \tag{4.7}$$

Note that, in the case considered, the solution of the cell problem depends on the initial stresses, since the principal coefficients in cell problem (4.6) are A_{ijkl}. The investigation of the cell problem carried out earlier in [11, 12] for monolithic composites (it differs from (4.6) in the boundary conditions) showed that, in general, the solutions of the cell problem depend in a complex non-linear way on the initial stresses.

Substituting (4.7) into (4.2) and taking into account the definitions of A_{ijkl} and b_{ijkl} we obtain

$$\sigma_{ij}^{(-4)} = (a_{ij\alpha 1}(\mathbf{y}) + \sigma^*_{j1}(x_1, \mathbf{y})\delta_{i\alpha} - a_{ij1\alpha}(\mathbf{y}) - \sigma^*_{j\alpha}(x_1, \mathbf{y})\delta_{i1})u^{(0)}_{\alpha,1y}(x_1) + \tag{4.8}$$
$$+(A_{ij11}(x_1, \mathbf{y}) + A_{ijkl}(x_1, \mathbf{y})N^{11}_{\sigma k,ly}(\mathbf{y}))u^{(0)}_{1,1y}(x_1) + (a_{ij\beta B}(x_1, \mathbf{y})s_\beta + \sigma^*_{jB}(x_1, \mathbf{y})s_\beta\delta_{i\beta})\phi(x_1)$$

Averaging (4.8) over the periodicity cell Y, and taking into account the symmetry of the elasticity constants [3], Remark 2 and Proposition 2, we obtain

$$N_{ij}^{(-4)} = \langle\sigma_{ij}^{(-4)}\rangle = (\langle\sigma^*_{j1}\rangle\delta_{i\alpha} - \langle\sigma^*_{j\alpha}\rangle\delta_{i1})u^{(0)}_{\alpha,1y}(x_1) + \tag{4.9}$$
$$+ \langle A_{ij11}(x_1, y) + A_{ijkl}(x_1, y)N^{11}_{\sigma k,ly}(y)\rangle u^{(0)}_{1,1y}(x_1) + \langle\sigma^*_{jB}\rangle s_\beta\delta_{i\beta}\phi(x_1)$$

When $ij = 11$, by virtue of Proposition 2 and conditions (1.3), the last formula in (4.9) takes the form

$$N_{11}^{(-4)} = \langle\sigma_{ij}^{(-4)}\rangle = A(\sigma)u^{(0)}_{1,1x}(x_1), \quad A(\sigma) = \langle a_{1111}(y) + A_{11kl}(x_1, \mathbf{y})N^{11}_{\sigma k,ly}(\mathbf{y})\rangle \tag{4.10}$$

Although in general there is no guarantee that the quantity $A(\sigma)$, which has the meaning of stiffness to tension, is positive, we can state the following sufficient condition for it to be positive

$$|\sigma^*_{jl}| \ll |a_{ijkl}| \tag{4.11}$$

Condition (4.11) is "physically natural", since the initial stresses cannot exceed the tensile strength of the beam material, which in turn does not exceed 0.01 of Young's modulus (and is often less) [3].

From the condition for $A(\sigma)$ to be positive, the equilibrium equation (3.1) and the boundary condition $u_1^{(0)}(-a) = u_1^{(0)}(a) = 0$, which follows from (2.1), we can conclude that $u_1^{(0)}(x_1) = 0$.

Hence, when condition (4.10) is satisfied, Eq. (4.7) takes the form

$$\mathbf{u}^{(1)} = -y_\alpha \mathbf{e}_1 u^{(0)}_{\alpha,1x}x_1 + y_B s_\beta \mathbf{e}_\beta \phi(x_1) + \mathbf{V}(x_1) \tag{4.12}$$

which is identical with the form of the function $\mathbf{u}^{(1)}$ in the problem for a stress-free beam [1].

Proposition 5. Equilibrium equations (3.2) are satisfied identically when conditions (1.3) and the conditions of Proposition 1 are satisfied.

To verify this consider the quantities in (3.2). Taking (4.12) into account, Eq. (4.8) gives

$$\sigma_{ij}^{(-4)} = (\sigma^*_{j1}(x_1, y)\delta_{i\alpha} - \sigma^*_{j\alpha}(x_1, y)\delta_{i1})u^{(0)}_{\alpha,1y}(x_1) + \sigma^*_{jB}(x_1, y)s_\beta\delta_{i\beta}\phi(x_1)$$

We integrate this equality over the periodicity cell Y. We also integrate this equality over the periodicity cell, first multiplying it by y_β, with $j = 1$. Taking into account Remark 2 and Proposition 2, we obtain

$$N_{ij}^{(-4)} = (\langle\sigma^*_{j1}\rangle\delta_{i\alpha} - \langle\sigma^*_{j\alpha}\rangle\delta_{i1})u^{(0)}_{\alpha,1y}(x_1) + \langle\sigma^*_{jB}\rangle s_\beta\delta_{i\beta}\phi(x_1) \tag{4.13}$$

$$M_{i\beta}^{(-4)} = (\langle\sigma^*_{j1}y_\beta\rangle\delta_{i\alpha} - \langle\sigma^*_{j\alpha}y_\beta\rangle\delta_{i1})u^{(0)}_{\alpha,1y}(x_1) + \langle\sigma^*_{jB}y_\beta\rangle s_\beta\delta_{i\beta}\phi(x_1)$$

Formula (2.5) with $m = -3$ yields

$$\sigma_{ij}^{(-3)} = A_{ijk1}(x_1, y)u^{(1)}_{k,1x} + A_{ijkl}(x_1, y)u^{(2)}_{k,ly}$$

Substituting expression (4.12) into it, we obtain

$$\sigma_{ij}^{(-3)} = -A_{ij11}(x_1, y)y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + A_{ij\beta 1}(x_1, y)y_B s_\beta \phi_{,1x}(x_1) + \tag{4.14}$$
$$+A_{ijk1}(x_1, y)V_{k,1x}(x_1) + A_{ijkl}(x_1, y)u^{(2)}_{k,ly}$$

Equilibrium equations (3.2) with $i = 1$ contain the undefined function $N_{1\alpha}^{(-3)}$. To eliminate it from (3.2) we consider the difference $N_{1\alpha}^{(-3)} - N_{1\alpha}^{(-3)}$, which, using (4.14) and the definition of A_{ijkl}, can be written in the form

$$N_{1\alpha}^{(-3)}-N_{\alpha 1}^{(-3)}=\left\langle\sigma_{1\alpha}^{(-3)}-\sigma_{\alpha 1}^{(-3)}\right\rangle=\left\langle-\left(\sigma_{\alpha 1}^{*}-\sigma_{11}^{*}\delta_{\alpha 1}\right)y_\beta\right\rangle u_{\beta,1x1x}^{(0)}(x_1)+$$

$$+\left\langle(\sigma_{\alpha 1}^{*}-\sigma_{11}^{*}\delta_{\alpha 1})y_B\right\rangle s_\beta\phi_{,1x}+\left\langle\sigma_{\alpha 1}^{*}\delta_{1k}-\sigma_{11}^{*}\delta_{\alpha 1}\right\rangle V_{k,1x} \tag{4.15}$$

The coefficients on the right-hand sides of the formulae for $N_{ij}^{(-4)}$, $M_{i\beta}^{(-4)}$ from (4.13) and in (4.15) are equal to zero by virtue of conditions (1.3) an Proposition 2. Consequently, Eqs (3.2) are satisfied identically when $i = 1$.

We will investigate the next terms of the asymptotic expansion. We now choose $m = -3$, $n = 1$ and $\mathbf{v} = \varepsilon\mathbf{v}(\mathbf{y})$ in (2.6). We then obtain the problem

$$\sigma_{ij,iy}^{(-3)}=0 \text{ in } G_1,\ \sigma_{ij}^{(-3)}n_j=0 \text{ on } \gamma_1 \tag{4.16}$$

Consider the next cell problems: the cell problem with respect to the functions $N_\sigma^{2\alpha}(y)$

$$\left(A_{ijkl}(\mathbf{y})N_{\sigma k,ly}^{2\alpha}-a_{ij11}(\mathbf{y})y_\alpha-\sigma_{j1}^{*}(x_1,\mathbf{y})\delta_{i1}y_\alpha\right)_{,ij}=0 \text{ in } Y$$

$$\left(A_{ijkl}(\mathbf{y})N_{\sigma k,ly}^{2\alpha}-a_{ij11}(\mathbf{y})y_\alpha-\sigma_{j1}^{*}(x_1,\mathbf{y})\delta_{i1}y_\alpha\right)n_j=0 \text{ on } \gamma \tag{4.17}$$

$N_\sigma^{2\alpha}(y)$ is periodic in y_1 with period m.
The cell problem with respect to the function $\mathbf{X}_\sigma^3(\mathbf{y})$ has the form

$$\left(A_{ijkl}(\mathbf{y})X_{\sigma k,ly}^{3}+a_{ij\beta 1}(\mathbf{y})y_B s_\beta+\sigma_{j1}^{*}(x_1,\mathbf{y})\delta_{i\beta}y_B s_\beta\right)_{,jy}=0 \text{ in } Y$$

$$\left(A_{ijkl}(\mathbf{y})X_{\sigma k,ly}^{3}+a_{ij\beta 1}(\mathbf{y})y_B s_\beta+\sigma_{j1}^{*}(x_1,\mathbf{y})\delta_{i\beta}y_B s_\beta\right)n_j=0 \text{ on } \gamma \tag{4.18}$$

$\mathbf{X}_\sigma^3(\mathbf{y})$ is periodic in y_1 with period m.
Then, taking Eqs (4.14) into account and using the functions $\mathbf{N}_\sigma^{11}$, $\mathbf{N}_\sigma^{2\alpha}$, $\mathbf{X}_\sigma^3$, the solution of (4.16) can be written in the form

$$\mathbf{u}^{(2)}=\mathbf{N}_\sigma^{2\alpha}(y)u_{\alpha,1x1x}^{(0)}(x_1)+\mathbf{X}_\sigma^3(\mathbf{y})\phi_{,1x}(x_1)+\mathbf{N}_\sigma^{11}(y)V_{1,1x}(x_1)-y_\alpha\mathbf{e}_1V_{\alpha,1x}(x_1) \tag{4.19}$$

and substituting (4.19) into (4.14) we obtain

$$\sigma_{ij}^{(-3)}=(A_{ij11}(x_1,\mathbf{y})y_\alpha+A_{ijkl}(x_1,\mathbf{y})N_{\sigma k,ly}^{2\alpha}(\mathbf{y}))u_{\alpha,1x1x}^{(0)}(x_1)+$$
$$+(A_{ij\beta 1}(x_1,\mathbf{y})y_B s_\beta+A_{ijkl}(x_1,\mathbf{y})X_{\sigma k,ly}^{3}(\mathbf{y}))\phi_{,1x}(x_1)+$$
$$+(A_{ij11}(x_1,\mathbf{y})+A_{ijkl}(x_1,\mathbf{y})N_{\sigma k,ly}^{11}(\mathbf{y}))V_{1,1x}(x_1)+$$
$$+(A_{ij\alpha 1}(x_1,\mathbf{y})+A_{ij1\alpha}(x_1,\mathbf{y}))V_{\alpha,1x}(x_1) \tag{4.20}$$

We integrate (4.20) over the periodicity cell Y. Taking the definitions of A_{ijkl} and b_{ijkl} and Proposition 2 into account, we obtain

$$N_{ij}^{(-3)}=A_{ij}^0V_{1,1x}+A_{ij\alpha}^1u_{\alpha,1x1x}^{(0)}+B_{ij}^0\phi_{,1x} \tag{4.21}$$

We multiply (4.19) by y_β with $j = 1$ and integrate the result over the periodicity cell Y. Taking the second condition from (1.3) into account we obtain

$$M_{i\beta}^{(-3)}={}^1A_{i\beta}V_{1,1x}+A_{i\beta\alpha}^2u_{\alpha,1x1x}^{(0)}+B_{i\beta}^1\phi_{,1x} \tag{4.22}$$

where

$$A_{ij}^0=\langle A_{ijk1}(x_1,\mathbf{y})+A_{ijkl}(x_1,\mathbf{y})N_{\sigma k,ly}^{11}(\mathbf{y})\rangle$$

$$A_{ij\alpha}^1=\langle A_{ij11}(x_1,\mathbf{y})y_\alpha+A_{ijkl}(x_1,\mathbf{y})N_{\sigma k,ly}^{2\alpha}(\mathbf{y})\rangle$$

$$\begin{aligned}
{}^{1}A_{i\beta} &=< (A_{ij11}(x_1,y)+A_{i1kl}(x_1,y)N^{11}_{\sigma k,ly}(y))y_\beta > \\
A^{2}_{i\beta\alpha} &=< (A_{ij11}(x_1,y)y_\alpha + A_{ijkl}(x_1,y)N^{2\alpha}_{\sigma k,ly}(y))y_\beta > \\
B^{0}_{ij} &=< A_{ij\beta 1}(x_1,y)y_B s_\beta + A_{ijkl}(x_1,y)X^{3}_{\sigma k,ly}(y) > \\
B^{1}_{i\beta} &=< (A_{i1\beta 1}(x_1,y)y_B s_\beta + A_{i1kl}(x_1,y)X^{3}_{\sigma k,ly}(y))y_\beta >
\end{aligned} \tag{4.23}$$

Formulae (4.21) and (4.22) are the equations of the stressed beam. The coefficients (4.23) of formulae (4.21) and (4.22) define the stiffness of the beam for extension, bending and torsion. Although formulae (4.23) are similar in form to the formulae for stress-free beams from [1], there is an important difference which is that the coefficients in (4.23) and the solutions of the cell problem depend on the initial stresses.

Using Propositions 2 and 3, formulae (4.23) can be written in the following form

$$\begin{aligned}
A^{0}_{ij} &=< a_{ijk1}(x_1,y) + a_{ijkl}(x_1,y)N^{11}_{\sigma k,ly}(y) > \\
A^{1}_{ij\alpha} &=< a_{ij11}(x_1,y)y_\alpha + a_{ijkl}(x_1,y)N^{2\alpha}_{\sigma k,ly}(y) > \\
{}^{1}A_{i\beta} &=< (a_{i111}(x_1,y)+a_{i1kl}(x_1,y)N^{11}_{\sigma k,ly}(y))y_\beta > + \\
&+ < \sigma^{*}_{1l}(x_1,y)N^{11}_{\sigma k,ly}(y)y_\beta > \delta_{ik} \\
A^{2}_{i\beta\alpha} &= < (a_{ij11}(x_1,y)y_\alpha + a_{ijkl}(x_1,y)N^{2\alpha}_{\sigma k,ly}(y))y_\beta > + < \sigma^{*}_{11}(x_1,y)y_\alpha y_\beta > \delta_{i1} \\
&+ < \sigma^{*}_{1l}(x_1,y)N^{2\alpha}_{\sigma k,ly}(y)y_\beta > \delta_{ik} \\
B^{0}_{ij} &= < A_{ij\beta 1}(x_1,y)y_B s_\beta + A_{ijkl}(x_1,y)X^{3}_{\sigma k,ly}(y) > + < \sigma^{*}_{jl}(x_1,y)X^{3}_{\sigma k,ly}(y) > \delta_{ik} \\
B^{1}_{i\beta} &= < (A_{i1\beta 1}(x_1,y)y_B s_\beta + A_{i1kl}(x_1,y)X^{3}_{\sigma k,ly}(y))y_\beta > + < \sigma^{*}_{11}(x_1,y)y_\gamma y_\beta > s_\gamma \delta_{i\gamma}
\end{aligned} \tag{4.24}$$

As can be seen, in general all the stiffness characteristics of the beam depend on the initial stresses.

5. SHEAR FORCES

In the asymptotic theory shearing forces play a role that differs from the axial forces and moments. This role is determined by their position in the equilibrium equations.

The equilibrium equations with $i = 1$ give, in particular

$$-M^{(-3)}_{1\alpha,1x} - N^{(-2)}_{1\alpha} = 0, \quad N^{(-2)}_{\alpha 1,1x} = 0 \tag{5.1}$$

The quantities $N^{(-2)}_{1\alpha}$ and $N^{(-2)}_{\alpha 1}$ in (5.1) have the meaning of hearing forces in the case of a classical uniform beam. We will retain this meaning for them in the case in question. If $N^{(-2)}_{ij}$ were symmetric for ij (like the stresses or forces are symmetric in the problem without initial stresses [3]), they could be eliminated in the usual way, namely, by differentiating the first equation in (5.1) and then using the second equation to eliminate $N^{(-2)}_{1\alpha}$. In the case considered there is no symmetry with respect to ij. We will proceed as follows. Noting that in expression (2.5) for $m = -2$ some of the terms are symmetric with respect to ij while some are not, we can write the equation

$$K_{ij} = N^{(-2)}_{ij} - N^{(-2)}_{ji} = < \sigma^{(-2)}_{ij} - \sigma^{(-2)}_{ji} > = < \left(b_{ijk1} - b_{jik1}\right)u^{(2)}_{k,1x} > + < \left(b_{ijkl} - b_{jikl}\right)u^{(3)}_{k,1x} >$$

By Proposition 3 the last term on the right-hand side is zero. Substituting (4.19) into this equation we obtain

$$\begin{aligned}
K_{ij} &= < \left(b_{ijk1} - b_{jik1}\right)N^{11}_{\sigma k} > V_{1,1x1x} - < \left(b_{ij11} - b_{ji11}\right)y_\alpha > V_{\alpha,1x1x} + \\
&+ < \left(b_{ijk1} - b_{jik1}\right)N^{2\alpha}_{\sigma k} > u^{(0)}_{\alpha,1x1x1x} + < \left(b_{ijk1} - b_{jik1}\right)X^{3}_{\sigma k} > \phi_{,1x1x}
\end{aligned} \tag{5.2}$$

By virtue of the second condition from (1.3)

$$<\left(b_{ij11}-b_{ji11}\right)y_\alpha> = <\sigma^*_{j1}y_\alpha>\delta_{i1} - <\sigma^*_{i1}y_\alpha>\delta_{j1} = 0$$

When $ij = 1\beta$ we obtain

$$K_{1\beta} = k_\beta V_{1,1x1x} + k_{\beta\alpha}u^{(0)}_{\alpha,1x1x1x} + l_\beta \phi_{,1x1x} \tag{5.3}$$

where, taking into account the definition of b_{ijkl}

$$\begin{aligned} k_\beta &= <\sigma^*_{\beta 1}N^{11}_{\sigma k}> - <\sigma^*_{11}N^{11}_{\sigma k}>\delta_{\beta k} \\ k_\beta &= <\sigma^*_{\beta 1}N^{11}_{\sigma k}> - <\sigma^*_{11}N^{11}_{\sigma k}>\delta_{\beta k} \\ l_\beta &= <\sigma^*_{\beta 1}X^3_{\sigma k}>\delta_{1k} - <\sigma^*_{11}X^3_{\sigma k}>\delta_{\beta k} \end{aligned} \tag{5.4}$$

The equation for the bending moments follows from (5.1) (it is obtained by differentiating the first equation of (5.1) and substituting $N^{(-2)}_{1\beta} = N^{(-2)}_{\beta 1} + K_{1\beta}$ using the second equation of (5.1))

$$-M^{(-3)}_{1\beta,1x1x} = K_{1\beta,1x} \tag{5.5}$$

The quantity $M^{(-3)}_{1\beta}$ is defined by (4.22) while $K_{1\beta}$ is the defined by (5.3).

Hence, (5.5) does not contain the strictly shear forces $N^{(-2)}_{ij}$, but only their antisymmetric part, given by (5.3). To calculate the shearing forces $N^{(-2)}_{ij}$ themselves, it is necessary to obtain the next terms of the asymptotic expansion. This is not required for calculating the anti-symmetric part of the shear forces.

For the torque $M = M^{(-2)}_{32} - M^{(-2)}_{23}$ we obtain from the first equation of (3.3) $-M_{,1x} = (N^{(-2)}_{32} - N^{(-2)}_{23})$. By (5.2) we have

$$K_{32} = kV_{1,1x1x} + k_\alpha u^{(0)}_{\alpha,1x1x1x} + l\phi_{,1x1x} \tag{5.6}$$

where we have taken into account the definition of b_{ijkl}

$$k = <\sigma^*_{21}N^{11}_{\sigma k}>\delta_{3k} - <\sigma^*_{31}N^{11}_{\sigma k}>\delta_{2k}, \quad k_\alpha = <\sigma^*_{21}N^{2\alpha}_{\sigma k}>\delta_{3k} - <\sigma^*_{31}N^{2\alpha}_{\sigma k}>\delta_{2k} \tag{5.7}$$

$$l = <\sigma^*_{21}X^3_{\sigma k}>\delta_{3k} - <\sigma^*_{31}X^3_{\sigma k}>\delta_{2k}$$

As a result we have the following equation for the torques

$$-M_{,1x} + K_{32} = 0 \tag{5.8}$$

The quantity M is defined by (4.22) and K_{32} is defined by (5.6). Note once again that Eq. (5.8) does not contain the shear forces themselves but only their antisymmetric part.

The solutions of the cell problem, as was pointed above, are determined apart from a constant. Moreover, the solution of cell problem (4.18) is determined apart from the function $y_\beta s_\beta \mathbf{e}_\beta \phi(x_1)$.

Proposition 6. The above arbitrariness in determining the solutions of the cell problem has no effect on the value of (5.4) and (5.7).

In (5.4) and (5.7) this arbitrariness leads to the occurrence of the terms $\langle \sigma^*_{i1} \rangle$ and $\langle \sigma^*_{\alpha 1} yB \rangle$. These terms are equal to zero in view of condition (1.3) and Proposition 2.

6. BOUNDARY CONDITIONS

When the beam is rigidly clamped, we obtain the following equations from the initial boundary condition $\mathbf{u}^\varepsilon(\mathbf{x}) = 0$ when $x_1 = -a, a$ and expansion (2.1) as in [1]

$$V_1(-a) = V_1(a) = u^{(0)}_\alpha(-a) = u^{(0)}_\alpha(a) = u^{(0)}_{\alpha,1x}(-a) = u^{(0)}_{\alpha,1x}(a) = \phi(-a) = \phi(a) = 0 \tag{6.1}$$

In the case of a hinge support we also obtain the classical boundary conditions.

Differences arise when considering the free edge. In this case, taking $n = 0, 1$ and the trial function $\mathbf{v}$ in the form $\mathbf{v}(x_1)$ and $\varepsilon[y_2\mathbf{v}_2(x_1) + y_3\mathbf{v}_3(x_1)]$, we obtain

$$N_{11}^{(-3)}(-a)=N_{11}^{(-3)}(a)=0,\ M_{1\beta}^{(-3)}(-a)=M_{1\beta}^{(-3)}(a)=0,\ M(-a)=M(a)=0 \tag{6.2}$$

$$N_{\alpha 1}^{(-2)}(-a)=N_{\alpha 1}^{(-2)}(a)=0 \tag{6.3}$$

Equation (6.3) is inconvenient to use since, to obtain an expression for $N_{\alpha 1}^{(-2)}$ in terms of the deformation characteristics, we must construct the subsequent terms of the expansion. We will proceed as follows. According to equilibrium equation (3.3) $-M_{1\alpha,1x}^{(-3)} + N_{1\alpha}^{(-2)} = 0$, including when $x_1 = -a, a$. Substituting $N_{1\alpha}^{(-2)} = N_{\alpha 1}^{(-2)} + K_{1\alpha}$ here we obtain $-M_{1\alpha,1x}^{(-3)} + K_{1\alpha} = 0$. When $x_1 = -a, a$ we obtain

$$-M_{1\alpha,1x}^{(-3)}+K_{1\alpha}=0 \ \text{ when } \ x_1=-a,a \tag{6.4}$$

Equation (6.4), which replaces (6.3), contains only the functions V_1, $u_\alpha^{(0)}$, ϕ.

7. CYLINDRICAL BEAM

Suppose we have a classical cylindrical beam of a homogeneous isotropic material, in which only an axial force acts. The latter means that $\sigma_{11}^* \neq 0$, $\sigma_{ij}^* = 0$ when $ij \neq 11$.

In the case considered, the functions a_{ijkl}, σ_{ij}^* are independent of y_1 and the solutions of the cell problem can be sought in the form of a function of the arguments y_2 and y_3 [1]. We have (Z denotes one of the functions—the solutions of the cell problem N_σ^{11}, $N_\sigma^{2\alpha}$ or X_σ^3) $A_{ijkl}Z_{k,ly} = A_{ijk\alpha}Z_{k,\alpha y} = a_{ijk\alpha}Z_{k,\alpha y}$ by virtue of the fact that $A_{ijk\alpha} = a_{ijk\alpha} + \sigma_{j\alpha}^*\delta_{ik}$ and $\sigma_{j\alpha}^* = 0$.

The free terms in the equations of the cell problem (4.6), (4.18), (4.19) are equal to zero by virtue of Proposition 12 and the fact that $\sigma_{ij}^* = 0$ when $ij \neq 11$ (this can be checked by differentiating them).

For the free terms in the boundary conditions of the same cell problem, by virtue of Proposition 1 and the fact that $n_1 = 0$ on γ and $\sigma_{\alpha 1}^* = 0$, we have $\sigma_{i1}^* n_j = 0$ on γ.

By virtue of this, the cell problem in the case considered is identical with the cell problem for a cylindrical stress-free beam, considered previously [1], and N_σ^{11}, $N_\sigma^{2\alpha}$, N_σ^3 in (4.24) can be replaced by N^{11}, $N^{2\alpha}$, X^3—the solutions of the cell problem for a stress-free beam (obtained from (4.6), (4.28) and (4.29) when $\sigma_{ij}^* = 0$; for more detail see [1]). Taking this into account, formulae (4.24) in the case considered can be written in the form

$$\begin{gathered}A_{ij}^0=A_{ij}^0(0),\ A_{ij\alpha}^1=A_{ij\alpha}^1(0),\ {}^1A_{i\beta}={}^1A_{i\beta}(0),\ B_{ij}^0=B_{ij}^0(0)\\ A_{i\beta\alpha}^2=A_{i\beta\alpha}^2(0)+\langle\sigma_{11}^*(x_1,y)y_\alpha y_\beta\rangle\delta_{i1}\\ B_{i\beta}^1=B_{i\beta}^1(0)+\langle\sigma_{11}^*(x_1,y)y_\Gamma y_\beta\rangle s_\gamma\delta_{i\gamma}\end{gathered} \tag{7.1}$$

The quantity Γ is defined in the same way as B. The argument (0) denotes the stiffness of the stress-free beam. According to (7.1), in the case considered, the initial stresses only affect the stiffness for bending and torsion.

Consider the stiffness for torsion. In the classical case it is defined as $B = B_{32}^1 - B_{23}^1$. We have the following expression for B

$$B=B(0)+\langle\sigma_{11}^*(x_1,y)(y_2^2+y_3^2)\rangle$$

In this formula the last term is found to be identical with the term which takes the initial stresses into account from [6].

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